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1992 J. Phys. A: Math. Gen. 25 2711

(http://iopscience.iop.org/0305-4470/25/9/038)

J. Phys. A: Math. Gen. 25 (1992) 2711–2720. Printed in the UK

Multiparameter solutions of the Yang–Baxter equation

Shao-Ming Fei†, Han-Ying Guo† and He Shi‡

† CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China, and (mailing address) Institute of Theoretical Physics, Academia Sinica, PO Box 2735, Beijing 100080, People's Republic of China

‡ Institute of Systems Science, Academia Sinica, Beijing 100080, People's Republic of China

Received 16 September 1991

Abstract. The multiparameter solutions, 4×4 R-matrices, of quantum Yang–Baxter equation have been systematically investigated and more than thirty selected solutions are presented by using Wu elimination. The free parameters and non-zero eigenvalues of these R-matrices vary from one to five and zero to four respectively. They are classified according to the corresponding algebraic structures.

1. Introduction

The quantum Yang–Baxter equation (YBE) [1] has been studied as the master equation in integrable models in statistical mechanics and quantum field theory in two dimensions for more than twenty years. Recent progress shows that it plays a profound role in a variety of diverse problems in theoretical physics such as exactly soluble models (such as the six- and eight-vertex models) in statistical mechanics [2], integrable model field theories [3], exact S-matrix theory [4], two-dimensional field theories involving fields with intermediate statistics [5], conformal field theory [6] and quantum groups [7], which have shed new light on the significance of this equation.

In this paper we study the quantum YBE without spectral parameters by using Wu elimination. Let V be a complex vector space and R the solution of the YBE. Then R takes values in $\mathrm{End}_C(V \otimes V)$ and satisfies the YBE

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{1}$$

where $\mathcal{R}_{ij}$ signifies the matrix on the complex vector space $V \otimes V \otimes V$, acting as $\mathcal{R}$ on the ith and the jth components and as the identity on the other component, e.g. $\mathcal{R}_{12} = \mathcal{R} \otimes I$.

Basically V is two-dimensional and R are 4×4 matrices. In this case $\mathcal{R}_{ij}$ are 8×8 matrices. Therefore the YBE (1) is a set of equations with 64 equations and 16 complex variables. To solve the set of equations is to find the zero sets of the 64 equations defined in 16-dimensional complex space. Higher rank matrix solutions can be constructed with these 4×4 matrix solutions.

Unfortunately little is known about the 4×4 matrix solution of the YBE. Originally what is known is the so called 'standard' solution R_q and its supersymmetric counterpart solution R'_q,

$$R_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad R'_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

With respect to the quantum group R_q gives rise to the q-deformed algebra $SU_q(2)$ and R'_q corresponds to the case of superalgebra. Both R_q and R'_q have only one parameter.

Recently a two-parameter solution of the YBE for R_q has been found [8]

$$R_{p,q} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-p^{-1} & q/p & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

which reduces to R_q when p approaches q.

It is natural to ask whether any other multiparameter solutions for such a significant equation can be found. That is, whether a system of algebraic equations consisting of 64 cubic polynomial equations can be well solved. Wu elimination, established by Wu Wen-tsün [9], provides a perfect method to solve such a system of algebraic equations. In this paper we present a series of selected multiparameter solutions of the YBE. The systematic approach to the YBE using Wu elimination will be given in [10].

2. The multiparameter solutions

For convenience we set the R-matrix to be

$$\begin{pmatrix} X_{11} & X_3 & X_1 & X_{17} \\ X_5 & X_{12} & X_{15} & X_2 \\ X_7 & X_{16} & X_{13} & X_4 \\ X_{18} & X_8 & X_6 & X_{14} \end{pmatrix} \tag{2}$$

where the X_i are complex variables. The YBE is invariant under certain similarity transformations. Let A be a 2×2 non-singular matrix. If R is a solution of the YBE, then

$$R' = (A \otimes A) R (A^{-1} \otimes A^{-1}) \tag{3}$$

is also a solution of the YBE. It is easy to find that there exists a X_i-dependent matrix A that transforms matrix (1) to a new one with matrix elements $R(1,2) = R(1,3)$ and $R(2,4) = R(3,4)$ or $R(2,1) = R(3,1)$ and $R(4,2) = R(4,3)$. What is more, from our experience, instead of matrix (2), it is safe to postulate that an R-matrix is of the form

$$\begin{pmatrix} X_{11} & X_1 & X_1 & X_{17} \\ X_5 & X_{12} & X_{15} & X_2 \\ X_5 & X_{16} & X_{13} & X_2 \\ X_{18} & X_6 & X_6 & X_{14} \end{pmatrix}. \tag{4}$$

So far all the solutions we have found are of the form (4) up to similarity transformation (3).

Substituting (4) into YBE (1) we obtain many solutions with a varying number of parameters and eigenvalues. In order to classify these solutions we work with the corresponding braid group representations.

Let R be a solution of the YBE and $\check{R} = RP$, where P is the permutation matrix, then $\check{R}$ satisfies

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}$$

where $\check{R}_{12} = \check{R} \otimes I$ and $\check{R}_{23} = I \otimes \check{R}$.

Investigating the Temperley–Lieb and Birman–Wenzl algebraic structures by use of the braid group representations, we find that four solutions are of Temperley–Lieb algebraic structures, eighteen solutions seem to be of quantum superalgebras; ten solutions are of Birman–Wenzl algebraic structures; and there are sixteen solutions possessing four different eigenvalues. The remaining solutions have zero eigenvalues. Namely, they have no inverses. Hence they are not meaningful.

Now we present part of the multiparameter solutions according to this classification. The systematic and complete solutions of the quantum YBE will be presented in future papers.

2.1. *Solutions of the YBE with Temperley–Lieb algebras*

First the solutions in form (4) include the solutions R_q and $R_{p,q}$

$$R_1 = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ 0 & X_{12} & 0 & 0 \\ 0 & \dfrac{X_{11}^2 - X_{13}X_{12}}{X_{11}} & X_{13} & 0 \\ 0 & 0 & 0 & X_{11} \end{pmatrix}.$$

The eigenvalues of R_1 are X_{11}, X_{12} and X_{13}. While the eigenvalues of $\check{R}_1$ are X_{11} and $-X_{12}X_{13}/X_{11}$ satisfying

$$(\check{R}_1 - X_{11} \cdot I^{(4)})^3 \left(\check{R}_1 + \frac{X_{12}X_{13}}{X_{11}} \cdot I^{(4)}\right) = 0$$

which can be reduced to

$$(\check{R}_1 - X_{11} \cdot I^{(4)}) \left(\check{R}_1 + \frac{X_{12}X_{13}}{X_{11}} \cdot I^{(4)}\right) = 0$$

where $I^{(4)}$ is a 4×4 identical matrix. Therefore the Templerly–Lieb algebra can be readily constructed. It is clear that if one takes $X_{11} = q$, $X_{12} = X_{13} = 1$, R_1 becomes the 'standard' solution R_q; and taking $X_{11} = q$, $X_{12} = 1$, $X_{13} = q/p$, one gets the two-parameter solution $R_{p,q}$. In fact R_1 is equivalent to $R_{p,q}$ because a solution may be multiplied by a constant.

Another example for solution with Temperley–Lieb algebra is

$$R_2 = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ 0 & -X_{11} & 0 & 0 \\ 0 & X_{12} + X_{11} & X_{12} & 0 \\ 0 & 0 & 0 & X_{11} \end{pmatrix}.$$

2.2. *Quantum superalgebraic type solutions of the* YBE

$$R_3 = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ 0 & X_{12} & 0 & 0 \\ 0 & X_{12}X_3 + X_{11} & -X_{11}X_3 & 0 \\ 0 & 0 & 0 & X_{12}X_3 \end{pmatrix}.$$

The eigenvalues of $\check{R}_3$ are still two, X_{11} and $X_{12}X_{13}$. But $\check{R}_3$ now satisfies the equation

$$(\check{R}_3 - X_{12}X_{13} \cdot I^{(4)})^2(\check{R}_3 - X_{11} \cdot I^{(4)})^2 = 0.$$

Hence although this equation reduces to

$$(\check{R}_3 - X_{12}X_{13} \cdot I^{(4)})(\check{R}_3 - X_{11} \cdot I^{(4)}) = 0$$

it is impossible to construct the Temperley–Lieb algebra. By calculating the equation $R_3T_1T_2 = T_2T_1R_3$, where $T_1 = T \otimes I$, $T_2 = I \otimes T$ and T is 2×2 matrix, one finds that R_3 corresponds to a superalgebra [11]. In fact, by taking $X_{11} = q$ and $X_3 = -q^{-1}$, R_3 becomes the solution R'_q which gives rise to a superalgebra.

More examples are in order:

$$R_4 = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ 0 & X_{11} & 0 & 0 \\ 0 & 0 & X_{11} & 0 \\ 0 & X_6 & X_6 & -X_{11} \end{pmatrix}$$

$$R_5 = \begin{pmatrix} X_{11} & 0 & 0 & X_{17} \\ 0 & X_{11} & X_{11} - X_{12} & 0 \\ 0 & 0 & X_{12} & 0 \\ 0 & 0 & 0 & -X_{12} \end{pmatrix}$$

$$R_6 = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ 0 & \dfrac{X_{11} \pm a}{2} & 0 & X_2 \\ 0 & 0 & \dfrac{X_{11} \pm a}{2} & X_2 \\ 0 & X_3X_2 & X_3X_2 & \mp a \end{pmatrix}$$

where $a = \sqrt{X_{11}^2 - 4X_3X_2^2}$.

$$R_7 = \begin{pmatrix} X_{11} & X_1 & X_1 & 0 \\ X_3X_1 & -X_{11} & \dfrac{X_3X_1^2}{X_{11}} & 0 \\ X_3X_1 & \dfrac{X_3X_1^2}{X_{11}} & -X_{11} & 0 \\ 0 & 0 & 0 & -X_{11} \end{pmatrix}$$

$$R_8 = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ -X_{11}X_3 & 0 & X_2X_3 & X_2 \\ -X_{11}X_3 & X_2X_3 & 0 & X_2 \\ 0 & 0 & 0 & X_{11} \end{pmatrix}$$

$$R_9 = \begin{pmatrix} \frac{X_6X_2X_3^2-2}{X_3} & X_2 & X_2 & X_2^2X_3 \\ X_6 & -X_6X_2X_3 & X_6X_2X_3 & X_2 \\ X_6 & X_6X_2X_3 & -X_6X_2X_3 & X_2 \\ X_6^2X_3 & X_6 & X_6 & \frac{X_6X_2X_3^2-2}{X_3} \end{pmatrix}$$

$$R_{10} = \begin{pmatrix} X_{11} & -X_2 & -X_2 & -\frac{1}{X_4} \\ -X_2X_4(2X_{11}+X_2^2X_4) & -X_{11} & X_2^2X_4 & X_2 \\ -X_2X_4(2X_{11}+X_2^2X_4) & X_2^2X_4 & -X_{11} & X_2 \\ X_2^2X_4^2(2X_{11}+X_2^2X_4) & -X_2^3X_4^2 & -X_2^3X_4^2 & -(X_{11}+2X_2^2X_4) \end{pmatrix}.$$

2.3. Solutions of the YBE with Birman–Wenzl algebras

$$R_{11} = \begin{pmatrix} X_{11} & 0 & 0 & X_{17} \\ 0 & -X_{11} & 0 & 0 \\ 0 & 0 & -X_{11} & 0 \\ 0 & 0 & 0 & X_{11} \end{pmatrix}.$$

Both R_{11} and $\check{R}_{11}$ have two eigenvalues $\pm X_{11}$ and $\check{R}_{11}$ satisfies

$$(\check{R}_{11} - X_{11}\cdot I^{(4)})^3(\check{R}_{11} + X_{11}\cdot I^{(4)}) = 0$$

which reduces to

$$(\check{R}_{11} - X_{11}\cdot I^{(4)})^2(\check{R}_{11} + X_{11}\cdot I^{(4)}) = 0.$$

Owing to the degeneracy for eigenvalue X_{11}, $\check{R}_{11}$ can only have one kind of Birman–Wenzl algebra representation:

$$R_{12} = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ X_5 & X_{12} & 0 & 0 \\ X_5 & 0 & X_{12} & 0 \\ \frac{2X_5^2}{X_{11}-X_{12}} & -X_5 & -X_5 & X_{11} \end{pmatrix}$$

Here $\check{R}_{12}$ has three different eigenvalues X_{11} and $\pm X_{12}$ satisfying

$$(\check{R}_{12} - X_{11}\cdot I^{(4)})^2(\check{R}_{12} + X_{12}\cdot I^{(4)})(\check{R}_{12} - X_{12}\cdot I^{(4)}) = 0$$

which reduces to

$$(\check{R}_{12} - X_{11}\cdot I^{(4)})(\check{R}_{12} + X_{12}\cdot I^{(4)})(\check{R}_{12} - X_{12}\cdot I^{(4)}) = 0.$$

Hence one can construct two classes of Birman–Wenzl algebra representations for $\check{R}_{12}$.

It is straightforward to check that the following two solutions have the same properties,

$$R_{13}=\begin{pmatrix} X_{11} & X_1 & X_1 & X_4X_1^2 \\ X_5 & \dfrac{X_{11}X_4-2}{X_4} & X_5X_4X_1 & -X_1 \\ X_5 & X_5X_4X_1 & \dfrac{X_{11}X_4-2}{X_4} & -X_1 \\ X_4X_5^2 & -X_5 & -X_5 & X_{11} \end{pmatrix}$$

$$R_{14}=\begin{pmatrix} X_{11} & X_1 & X_1 & X_{17} \\ X_5 & -X_{11} & X_{15} & -X_1 \\ X_5 & X_{15} & -X_{11} & -X_1 \\ \dfrac{X_{17}X_5^2+X_{15}^2X_{11}-2X_{15}X_5X_1}{X_{17}X_{11}-X_1^2} & -X_5 & -X_5 & X_{11} \end{pmatrix}.$$

For solutions of the YBE in which their braid group representation matrices have four different eigenvalues, the algebraic structures need further investigation. Here are some solutions of this kind.

$$R_{15}=\begin{pmatrix} X_{11} & X_1 & X_1 & 0 \\ X_3X_1 & \dfrac{-X_{11}\pm b}{2} & 0 & 0 \\ X_3X_1 & 0 & \dfrac{-X_{11}\pm b}{2} & 0 \\ 0 & 0 & 0 & \pm b \end{pmatrix}$$

where $b=\sqrt{X_{11}^2+4X_3X_1^2}$;

$$R_{16}=\begin{pmatrix} X_{11} & X_1 & X_1 & 0 \\ X_5 & X_{12} & 0 & X_1 \\ X_5 & 0 & X_{12} & X_1 \\ 0 & X_5 & X_5 & 2X_{12}-X_{11} \end{pmatrix}$$

$$R_{17}=\begin{pmatrix} \dfrac{X_{17}X_{12}-X_2X_1+X_1^2}{X_{17}} & X_1 & X_1 & X_{17} \\ 0 & X_{12} & 0 & X_2 \\ 0 & 0 & X_{12} & X_2 \\ 0 & 0 & 0 & \dfrac{X_{17}X_{12}-X_2X_1+X_2^2}{X_{17}} \end{pmatrix}$$

$$R_{18}=\begin{pmatrix} 4\mathrm{i}X_2X_3 & 2X_2 & 2X_2 & 0 \\ 2X_2X_3^2 & -\mathrm{i}X_2X_3 & 0 & X_2 \\ 2X_2X_3^2 & 0 & -\mathrm{i}X_2X_3 & X_2 \\ 0 & X_2X_3^2 & X_2X_3^2 & -\mathrm{i}2X_2X_3 \end{pmatrix}$$

where $\mathrm{i} = \sqrt{-1}$.

$$R_{19} = \begin{pmatrix} X_5X_3X_2 & -X_2 & -X_2 & X_2^2X_3 \\ X_5 & -X_5X_3X_2 & \dfrac{-1 \pm c}{X_3} & X_2 \\ X_5 & \dfrac{-1 \pm c}{X_3} & -X_5X_3X_2 & X_2 \\ X_5X_3 & -X_5 & -X_5 & X_5X_3X_2 \end{pmatrix}$$

where $c = \sqrt{X_2^2X_3^4X_5 - X_2X_3^2X_5 - X_2X_3^2 + 1}$;

$$R_{20} = \begin{pmatrix} \dfrac{X_5 - d \pm e}{2X_3} & X_1 & X_1 & -\dfrac{2X_2X_1X_3}{X_5} \\ X_5 & -X_3(X_2 - X_1) & \dfrac{\pm e - X_5}{2X_3} & X_2 \\ X_5 & \dfrac{\pm e - X_5}{2X_3} & -X_3(X_2 - X_1) & X_2 \\ 2X_5X_3 & -X_5 & -X_5 & \dfrac{X_5 + d \pm e}{2X_3} \end{pmatrix}$$

where $d = 2X_3^2(X_1 + X_2)$ and $e = \sqrt{X_5^2 - 4X_5X_3^2(X_2 - X_1) - 16X_2X_1X_3^4}$.

The YBE also contains many solutions with zero eigenvalues. For example,

$$R_{21} = \begin{pmatrix} \dfrac{1}{X_3} & -\dfrac{1}{X_3^2} & -\dfrac{1}{X_3^2} & \dfrac{1}{X_3^3} \\ 1 & -\dfrac{1}{X_3} & -\dfrac{1}{X_3} & \dfrac{1}{X_3^2} \\ 1 & -\dfrac{1}{X_3} & -\dfrac{1}{X_3} & \dfrac{1}{X_3^2} \\ X_3 & -1 & -1 & \dfrac{1}{X_3} \end{pmatrix}.$$

Both R_{21} and $\check{R}_{21}$ have eigenvalue zero with degeneracy four! The following solutions (for both R_i and $\check{R}_i$) also have zero eigenvalues.

$$R_{22} = \begin{pmatrix} X_{11} & \pm\dfrac{\mathrm{i}(X_{12} + X_{11})}{2} & \pm\dfrac{\mathrm{i}(X_{12} + X_{11})}{2} & -X_{12} \\ \pm\dfrac{\mathrm{i}(X_{12} + X_{11})}{2} & X_{12} & -X_{12} & \pm\dfrac{\mathrm{i}(X_{12} + X_{11})}{2} \\ \pm\dfrac{\mathrm{i}(X_{12} + X_{11})}{2} & -X_{12} & X_{12} & \pm\dfrac{\mathrm{i}(X_{12} + X_{11})}{2} \\ -X_{12} & \pm\dfrac{\mathrm{i}(X_{12} + X_{11})}{2} & \pm\dfrac{\mathrm{i}(X_{12} + X_{11})}{2} & -2X_{12} - X_{11} \end{pmatrix}$$

$$R_{23} = \begin{pmatrix} -X_5X_3 & -X_5X_3^2 & -X_5X_3^2 & -X_5X_3^3 \\ X_5 & X_5X_3 & X_5X_3 & X_5X_3^2 \\ X_5 & X_5X_3 & X_5X_3 & X_5X_3^2 \\ X_{18} & -X_5 & -X_5 & -X_5X_3 \end{pmatrix}$$

$$R_{24} = \begin{pmatrix} \frac{X_5X_4^2X_1+2}{X_4} & X_1 & X_1 & X_4X_1^2 \\ X_5 & X_5X_4X_1 & X_5X_4X_1 & -X_1 \\ X_5 & X_5X_4X_1 & X_5X_4X_1 & -X_1 \\ X_4X_5^2 & -X_5 & -X_5 & \frac{X_5X_4^2X_1+2}{X_4} \end{pmatrix}$$

$$R_{25} = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ X_5 & 0 & 0 & 0 \\ X_5 & 0 & 0 & 0 \\ -\frac{X_5X_6-X_5^2}{X_{11}} & X_6 & X_6 & -\frac{X_{11}X_6}{X_5} \end{pmatrix}$$

$$R_{26} = \begin{pmatrix} 0 & X_1 & X_1 & X_{17} \\ 0 & 0 & -\frac{X_1^2}{X_{17}} & 0 \\ 0 & -\frac{X_1^2}{X_{17}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{X_1^2}{X_{17}} \end{pmatrix}$$

$$R_{27} = \begin{pmatrix} 0 & X_1 & X_1 & -\frac{2X_1^2}{X_{12}} \\ 0 & X_{12} & 0 & -X_1 \\ 0 & 0 & X_{12} & -X_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These solutions are not meaningful. However, with respect to the braid group representations, they constitute semi-braid groups with one kind of group operation and certainly give rise to a class of links.

2.4. Other solutions

For solutions that do not seem to be given in the ansatz (4) we consider

$$R'_{28} = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ X_{11}-X_{12} & X_{12} & 0 & 0 \\ X_{12}-X_{11} & X_{11}-X_{12} & X_{11} & 0 \\ X_{12}-X_{11} & X_{11}+X_{12} & X_{11}+X_{12} & -X_{12} \end{pmatrix}$$

It is easy to find that R'_{28} is equivalent to a solution in ansatz (4), R_{28},

$$R_{28} = (A^{-1}\otimes A^{-1})R'_{28}(A\otimes A) = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ 0 & X_{12} & 0 & 0 \\ 0 & X_{11}-X_{12} & X_{11} & 0 \\ 0 & 0 & 0 & -X_{12} \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Similarly, the solution

$$R'_{29} = \begin{pmatrix} X_{11} & X_6 & X_2 & X_{15} \\ X_6 & X_{11} & X_{15} & X_2 \\ X_2 & X_{15} & X_{11} & X_6 \\ X_{15} & X_2 & X_6 & X_{11} \end{pmatrix}$$

can be transformed into

$$R_{29} = (B \otimes B) R'_{29} (B^{-1} \otimes B^{-1})$$
$$= \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & -(X_2 - X_{11} - X_6 + X_{15}) & 0 & 0 \\ 0 & 0 & X_2 + X_{11} - X_6 - X_{15} & 0 \\ 0 & 0 & 0 & g \end{pmatrix}$$

where $f = -(X_2 - X_{11} + X_6 - X_{15})$, $g = X_2 + X_{11} + X_6 + X_{15}$ and $B = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

The following two solutions can also be transformed into the form of ansatz (4) by non-singular transformation matrices depending on the values of X_i.

$$R_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ X_5(1 - X_1) & X_1 & 0 & 0 \\ X_5 X_3 (1 - X_1) & 1 + X_1 X_3 & -X_3 & 0 \\ X_5^2 (X_3 - X_1) & X_5(1 + X_1) & -X_5 X_3 (1 + X_1) & X_1 X_3 \end{pmatrix}$$

$$R_{31} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ X_5(1 - X_1) & X_1 & 0 & 0 \\ X_5 X_3 (1 - X_1) & 1 + X_1 X_3 & -X_3 & 0 \\ X_5^2 (1 - X_1)(1 + X_3) & X_5 X_1 (1 + X_3) & -X_5 (1 + X_3) & 1 \end{pmatrix}.$$

It is also easy to find that when $X_5 = 1$, R_{30} and R_{31} are equivalent to R_3 and R_1. Taking $X_{12} = X_1 X_{11}$ and $X_{13} = X_3$ we have

$$R_{30}(X_5 = 1) = (C \otimes C) R_3 (C^{-1} \otimes C^{-1}).$$

Let $X_{12} = X_{11} X_1$ and $X_{13} = -X_{11} X_3$ we get

$$R_{31}(X_5 = 1) = (C \otimes C) R_1 (C^{-1} \otimes C^{-1}).$$

Here the transformation matrix $C = \begin{pmatrix} X_{11}^{-1/4} & 0 \\ X_{11}^{-1/4} & X_{11}^{-1/4} \end{pmatrix}$.

3. Conclusion

It should be noted that the solutions given in this paper have a varying number of eigenvalues and parameters. There are as many as five (four which are indenpendent as the solution may be multiplied by an arbitrary constant) free parameters in the 4×4 matrix solution R_{14}. All these parameters are contained in the eigenvalues of R_{14} and $\check{R}_{14}$.

The Temperley–Lieb and Birman–Wenzl algebraic structures, link polynomials, integrable models, quantum groups and quantum algebras [7] as well as their classical realizations and geometric meanings [12] for every solution remain to be investigated further. It will also be interesting to study the Yang–Baxterization of the solutions and relate these constant solutions to the well-known hard-hexagon [2] and 'free-fermion' [13] R-matrices. The detailed construction of the Temperley–Lieb algebra and the Yang–Baxterization of some of these solutions are presented in [14].

Acknowledgments

The authors would like to thank Professor Z M Qiu, S K Wang and K Wu, Drs Y Q Li, L Y Wang and J Zhang for helpful discussions. The authors (S M Fei and H Shi) are grateful to CCAST (World Laboratory) for its hospitality and support. The work was finished during the CCAST Summer Institute on Quantum Groups and Field Theory in Lower Dimensions co-sponsored by ITP and NSFC. Work supported in part by the National Natural Science Foundation of China and CCAST (World Laboratory).

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